

Offset-Free Robust MPC of Systems with Mixed Stochastic and Deterministic Uncertainty

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Abstract: This paper presents a robust model predictive control (MPC) approach for offset-free tracking of piece-wise constant references in the presence of bounded deterministic and stochastic disturbances. The system is considered to be linear with two sources of additive bounded uncertainties on the states. The first uncertainty source accounts for unknown, deterministic structural/parametric plant-model mismatch. The second uncertainty source represents stochastic exogenous system disturbances. The proposed deterministic-stochastic robust MPC approach uses estimates of the deterministic model uncertainties to modify the nominal state and input targets. This allows for achieving offset-free tracking of the mean of the controlled variables. A non-conservative constraint tightening procedure is used to handle probabilistic state constraints and hard input constraints in the presence of stochastic uncertainties. The computational complexity of the proposed robust MPC approach is comparable to that of nominal MPC. The closed-loop performance of the proposed robust MPC approach is compared to that of robust tube-based MPC and stochastic MPC in a simulation study.

Keywords: Model predictive control, Offset-free tracking, and Probabilistic/robust tubes

1. INTRODUCTION

The ability to systematically cope with multivariable system dynamics, constraints, and conflicting control objectives has made model predictive control (MPC) an attractive control approach in a wide range of engineering applications, ranging from path planning and robotics applications to complex process systems [Morari and Lee, 1999, Mayne, 2014]. A key challenge in MPC arises from model uncertainty (a.k.a. plant-model mismatch). Even though MPC typically exhibits some degree of robustness to sufficiently small model uncertainty due to its receding-horizon implementation, marginal robust performance may not be satisfactory in many practical applications. This consideration has led to development of various (deterministic) robust MPC and stochastic MPC approaches that generally aim to guarantee robust stability and performance of the closed-loop system by, respectively, incorporating deterministic and stochastic descriptions of system uncertainties into the optimal control problem (e.g., see reviews [Bemporad and Morari, 1999, Mesbah, 2016]).

This work addresses the MPC for linear systems with mixed deterministic and bounded stochastic uncertainties. The key notion of this work is to distinguish between model uncertainty, which is generally deterministic, and exogenous disturbances that are often of (bounded) probabilistic nature in real systems. In practical control applications, control-oriented models are commonly derived from high-fidelity (first-principles) models that cannot be directly used in optimization-based control algorithms due to computational considerations. High-fidelity models can

be used to characterize deterministic uncertainty bounds for control-oriented models. Alternatively, system identification methods can be used for characterizing the deterministic model uncertainty (as well as stochastic disturbances) when control-oriented models are identified from input-output data [Ljung, 1998].

This paper presents a robust MPC approach that can systematically handle a mixture of bounded deterministic and stochastic uncertainties. This will allow for MPC to non-conservatively handle the original sources of system uncertainties. Concepts from robust tube-based [Mayne et al., 2005] and stochastic tube-based MPC [Kouvaritakis et al., 2010] are used to derive a computationally tractable MPC formulation, with computational complexity comparable to that of nominal MPC. The estimates of (unknown) plant-model mismatch are incorporated into the MPC problem to ensure offset-free tracking of the mean of the controlled variables. The closed-loop performance of the proposed deterministic-stochastic robust MPC approach is compared to that of a robust tube-based MPC approach [Alvarado et al., 2007] and a stochastic MPC approach [Lorenzen et al., 2015].

Notation. \mathbb{N} denotes the set of positive integers and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Given two sets X and Y such that $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$, the Minkowski set addition and the Pontryagin set difference are defined by $X \oplus Y = \{x + y : x \in X, y \in Y\}$ and $X \ominus Y = \{x : x \oplus Y \subseteq X\}$, respectively. The positive (semi)definite matrix A is denoted by $A > 0$ ($A \geq 0$). $\|x\|_A^2 = x^\top A x$ is the weighted 2-norm, $\rho(A)$ is the spectral radius, and $(a, b) = [a^\top, b^\top]^\top$.

2. PROBLEM STATEMENT

Consider an uncertain system described by the discrete-time, linear time-invariant representation

$$x_{k+1} = Ax_k + Bu_k + d_k + w_k, \quad (1a)$$

$$y_k = Cx_k + Du_k, \quad (1b)$$

with states $x_k \in \mathbb{R}^n$, control inputs $u_k \in \mathbb{R}^m$, unknown deterministic disturbances $d_k \in \mathbb{D} \subset \mathbb{R}^{m_d}$, stochastic disturbances $w_k \in \mathbb{W} \subset \mathbb{R}^{m_w}$, and outputs $y_k \in \mathbb{R}^p$. The matrices (A, B, C, D) and the convex, bounded sets \mathbb{D} and \mathbb{W} are assumed known. It is assumed that w_k for all $k \in \mathbb{N}$ are independent realizations of a real-valued random variable W , with realizations inside of \mathbb{W} . Moreover, W is assumed to have zero mean (without loss of generality) and finite variance with a known distribution.

We briefly elaborate on the generality of the system model (1). The deterministic disturbances d_k represent structural/parametric plant-model mismatch (and possibly unmeasured persistent system disturbances). For example, consider a system of the form $x_{k+1} = (A + A_\delta)x_k + (B + B_\delta)u_k + w_k$, where the pair (A_δ, B_δ) denotes unknown system perturbations. In this case, the input-output data from the system can be used to infer the deterministic disturbances given by $d_k = A_\delta x_k + B_\delta u_k$, which is in fact a systematic plant-model mismatch. Another example is when the system dynamics are described by a (high-fidelity) nonlinear model $x_{k+1} = f(x_k, u_k) + w_k$, wherein d_k would be defined as $f(x_k, u_k) - Ax_k - Bu_k$. This implies that the deterministic disturbance set \mathbb{D} can generally be obtained from either input-output system data, or from a high-fidelity system model. On the other hand, the stochastic disturbances w_k represent the effect of exogenous disturbances acting on the system. In practical control applications, the distribution of W is typically estimated from input-output data using standard methods (see [Ljung, 1998] for methods and further details).

The system (1) is subject to individual chance constraints on the states and hard constraints on the inputs

$$\mathbb{P}[h_j^\top x_k \leq f_j] \geq 1 - \epsilon_j, \quad j = 1, \dots, r, \quad k \in \mathbb{N}_0, \quad (2a)$$

$$Gu_k \leq g, \quad k \in \mathbb{N}_0, \quad (2b)$$

where $\mathbb{P}[v \in V]$ is the probability of event $v \in V$, $\epsilon_j \in [0, 1]$ is the allowed probability of constraint violation, and r is the number of state constraints. The parameter $h_j^\top \in \mathbb{R}^{1 \times n}$ denotes the j th row of $H \in \mathbb{R}^{r \times n}$, while f_j is the j th element of $f \in \mathbb{R}^r$. Let \mathbb{X} and \mathbb{U} denote the collection of state and input constraints, respectively.

Let $s_k \in \mathbb{R}^p$ be an asymptotically constant reference signal. The objective of this work is to design a recursively-feasible robust MPC controller that stabilizes the true system and steers the outputs to a neighborhood around s_∞ in the face of deterministic and stochastic disturbances in (1). When the deterministic disturbances d_k tend to steady state d_∞ , the system outputs should track the setpoint without offset.

Three key challenges will be addressed toward developing a deterministic-stochastic framework for robust MPC: (i) how to efficiently handle the constraints (2) so that they are guaranteed to hold during closed-loop operation, (ii) how to minimize the tracking error (a.k.a. offset) when d_k tends to a constant value d_∞ , and (iii) how to ensure the

feasibility of the MPC problem for all possible disturbance realizations when the optimization is initially feasible.

3. PRELIMINARY RESULTS

This section summarizes the key results from the robust and stochastic MPC literature used to solve the combined deterministic-stochastic robust MPC problem.

3.1 Dual Mode Prediction Method

We aim to regulate the following nominal system, which is obtained from (1) by neglecting d_k and w_k

$$z_{k+1} = Az_k + Bv_k, \quad (3)$$

where $z_k \in \mathbb{R}^n$ are the nominal states and $v_k \in \mathbb{R}^m$ are the nominal inputs. To counteract the effect of disturbances, the control inputs u_k are defined in terms of control actions c_k and a static feedback gain K , i.e.,

$$u_k = Kx_k + c_k, \quad (4)$$

where K is chosen such that $\Phi = A + BK$ is strictly stable ($\rho(\Phi) < 1$). Similarly, the nominal inputs are

$$v_k = Kz_k + c_k. \quad (5)$$

Let $\delta_k = d_k + w_k$ be the mixed deterministic and stochastic disturbance variables that must lie in a bounded set $\Delta = \mathbb{D} \oplus \mathbb{W}$. Under the control laws (4) and (5), the uncertain predictions of (1a) and the nominal predictions of (3) evolve as follows

$$x_{i|k} = z_{i|k} + e_{i|k}, \quad (6a)$$

$$z_{i+1|k} = \Phi z_{i|k} + Bc_{i|k}, \quad (6b)$$

$$e_{i+1|k} = \Phi e_{i|k} + \delta_{i|k}, \quad (6c)$$

with initial conditions $z_{0|k} = x_k$ and $e_{0|k} = 0$ ($i \in \mathbb{N}_0$). The free variables $c_{i|k}$ will be optimized online by the proposed MPC method. We use a dual mode prediction method in order to work with a finite number of decision variables. Mode 1 treats $c_{i|k}$, $i = 0, \dots, N - 1$ as decision variables, while mode 2 sets $c_{i|k} = \bar{u}_k - K\bar{x}_k$, $i \geq N$ for some finite horizon N where (\bar{x}_k, \bar{u}_k) denotes the desired steady state of the nominal system defined next.

3.2 Target Calculation and Regulator

As disturbances d_k and w_k are unmeasured, the steady-state condition is first defined with respect to the nominal system (3), which at time k must satisfy

$$\min \bar{u}_k^\top \bar{R} \bar{u}_k, \quad \text{s.t.}: \quad \begin{bmatrix} A - I & B \\ C & D \end{bmatrix} \begin{bmatrix} \bar{x}_k \\ \bar{u}_k \end{bmatrix} = \begin{bmatrix} 0 \\ s_k \end{bmatrix}. \quad (7)$$

The target tracking problem (7) has a unique solution whenever the following condition is satisfied

$$\text{rank} \begin{bmatrix} A - I & B \\ C & D \end{bmatrix} = n + m$$

and $p \leq m$, indicating that the matrix is full row rank [Pannocchia and Rawlings, 2003]. Methods for modifying the steady-state condition to achieve offset-free control in the presence of disturbances are discussed later. Constraints are not directly accounted for in (7) as they are handled in the regulator presented in the next section. The solution of (7) can be parametrized as [Limón et al., 2008]

$$(\bar{x}_k, \bar{u}_k) = M_\theta \theta_k, \quad s_k = N_\theta \theta_k, \quad (8)$$

where $\theta_k \in \mathbb{R}^{n_\theta}$ is a parameter vector that characterizes the unique solution, and M_θ and N_θ are suitable matrices.

3.3 Constraint Tightening Methods

Two methods for efficiently handling constraints (2) offline are discussed in this section. The first method uses the well-known deterministic tubes to ensure that (2) holds with probability one. The second method uses a combination of robust and probabilistic tubes to bound the evolution of the mixed deterministic and stochastic variable δ into the future.

Robust Tube of Trajectories. A simple, but conservative, approach for dealing with constraints (2) is to use tubes to bound the control error $e_i = (x_i - z_i)$ with dynamics $e_{i+1} = \Phi e_i + \delta_i$. When $\rho(\Phi) < 1$, there exists a robust positively invariant set E that satisfies

$$\Phi E \oplus \Delta \subseteq E. \quad (9)$$

Note that $e_0 = (x_0 - z_0) \in E$ implies $e_i \in E$, $\forall i \geq N$, which leads to Proposition 1.

Proposition 1. [Mayne et al., 2005]. Suppose $E \subset \mathbb{X}$, $KE \subset \mathbb{U}$, and the system states in (1) and (3) lie in \mathbb{X} and satisfy $e_0 = (x_0 - z_0) \in E$. If the nominal states and control inputs in (3) satisfy the tighter constraints $z_i \in \mathbb{X} \ominus E$ and $v_i \in \mathbb{U} \ominus KE$, respectively, then the states x_i and control inputs $u_i = K(x_i - z_i) + v_i$ of the controlled system (1) will satisfy the original constraints $x_i \in \mathbb{X}$ and $u_i \in \mathbb{U}$ for all $i \in \mathbb{N}_0$ and for all disturbance realizations.

Using Proposition 1, it can be guaranteed that (2) will hold for all future predictions (starting from the current time k) whenever

$$z_{i|k} \in \mathbb{X} \ominus E, \quad v_{i|k} \in \mathbb{U} \ominus KE, \quad i \in \mathbb{N}_0, \quad (10)$$

for any $x_k \in z_{0|k} \oplus E$. Clearly, this constraint tightening method is conservative since constraints (2) are satisfied for all possible realizations of stochastic disturbances w_k . The conservatism in constraint handling can be reduced by accounting for the distribution of stochastic disturbances and the admissible constraint violation probability ϵ_j .

Probabilistic Tubes. A method similar to that in [Kouvaritakis et al., 2010] can also be used for direct constraint tightening. To this end, we first establish a worst-case tightening for d_k , followed by a probabilistic tightening for w_k . The necessary and sufficient conditions for fulfilling the chance constraint (2a) are given below.

Proposition 2. At time k , predictions of (1) satisfy the chance constraints (2a) if and only if the nominal system (6) satisfies constraints

$$Hz_{i|k} \leq f - \eta_i, \quad i \in \mathbb{N}, \quad (11)$$

with $\eta_i = \eta_i^d + \eta_i^w$, where the j th element of η_i^d is the maximum value of the expression

$$h_j^\top (\Phi^{i-1} d_k + \dots + d_{k+i-1})$$

and the j th element of η_i^w is the minimum η such that

$$\mathbb{P}[h_j^\top (\Phi^{i-1} w_k + \dots + w_{k+i-1}) \leq \eta] = 1 - \epsilon_j.$$

Proof: At any time k , the predicted error (6c) satisfies

$$e_{i|k} = \Phi^{i-1} \delta_{0|k} + \dots + \delta_{i-1|k}, \quad (12)$$

where $e_{0|k} = 0$. Splitting $e_{i|k} = e_{i|k}^d + e_{i|k}^w$ into deterministic and stochastic error sources e^d and e^w , respectively, yields

$$e_{i|k}^d = \Phi^{i-1} d_k + \dots + d_{k+i-1}, \quad (13a)$$

$$e_{i|k}^w = \Phi^{i-1} w_k + \dots + w_{k+i-1}. \quad (13b)$$

Hence, (2a) implies $h_j^\top x_{i|k} = h_j^\top z_{i|k} + h_j^\top e_{i|k}^d + h_j^\top e_{i|k}^w$. From the definitions of η_i^d , η_i^w , and (13), it directly follows that (11) ensures $\mathbb{P}[h_j^\top x_{i|k} \leq f_j] \geq 1 - \epsilon_j$ for all $j = 1, \dots, r$, $i \in \mathbb{N}$, and all disturbance realizations. \square

By definition, η_i^d is contained within the set $H \bigoplus_{k=0}^i \Phi^k \mathbb{D}$. Since $\rho(\Phi) < 1$, it follows that the sequence of η_i^d is monotonically increasing and converges to $\lim_{i \rightarrow \infty} \eta_i^d = \bar{\eta}^d \in HF_\infty$, where $F_\infty = \bigoplus_{i=0}^\infty \Phi^i \mathbb{D}$ is the minimal robust positively invariant (RPI) set. Existing methods can be used for determining an RPI set that contains the minimal RPI set in finite time (see, e.g., [Rakovic et al., 2005]). Therefore, bounds on η_i^d can be found offline and used in the terminal set calculation as shown in the next section.

For any $i \in \mathbb{N}$, the vector η_i^w is the solution to r one-dimensional linear chance constrained optimization problems. Exact computation requires calculation of the distribution function of $h_j^\top (\Phi^{i-1} w_k + \dots + w_{k+i-1})$, which requires the evaluation of a multivariate convolution integral. As shown in [Lorenzen et al., 2015], an alternative efficient approach for solving these programs is to use the scenario approach, which has been used to solve chance constrained MPC problems [Calafiore and Fagiano, 2013]. The key advantage of the scenario approach is that specific guarantees on the probability of constraint violation can be provided by taking a sufficiently large number of samples from W .

Bounds on η_i^w can be derived using Chebyshev's inequality.

Corollary 1. [Kouvaritakis et al., 2010]. Let $\eta_{i,j}^w$ denote the j th element of η_i^w . For every $i \in \mathbb{N}$, $\eta_{i,j}^w$ satisfies

$$\eta_{i,j}^w \leq \kappa (h_j^\top P_i h_j)^{1/2}, \quad \kappa^2 = (1 - \epsilon_j) / \epsilon_j, \quad (14)$$

where $P_{i+1} = \Phi P_i \Phi + \mathbb{E}[WW^\top]$ with $P_1 = \mathbb{E}[WW^\top]$.

Although the input constraints (2b) can be tightened to ensure robust satisfaction, it would be less conservative to perform stochastic constraints tightening on (2b) in a similar manner to the state constraints [Lorenzen et al., 2015]. This is due to the fact that the optimal inputs are recomputed at every k and are adapted to account for the (observed) disturbance realizations. These probabilistic constraints are only enforced on the future predictions of the inputs while hard input constraints will still be guaranteed by the MPC controller since $u_k = v_{0|k} \in \mathbb{U}$ as a result of x_k being known exactly.

Let $\epsilon_u \in [0, 1]$. Similar to the state constraint tightening in Proposition 2, we can replace (2b) with

$$Gv_{i|k} \leq g - \mu_i, \quad i \in \mathbb{N}_0, \quad (15)$$

where $\mu_i = \mu_i^d + \mu_i^w$ and μ_i^d , μ_i^w are defined similarly to η_i^d , η_i^w using the rows of G , elements of g , and the allowed constraint violation probability ϵ_u . Note that $\mu_0 = 0$ as a result of $e_{0|k} = 0$ for every $k \in \mathbb{N}_0$.

4. DETERMINISTIC-STOCHASTIC ROBUST MPC FOR OFFSET-FREE TRACKING

This section presents a computationally tractable approach for the robust MPC for the linear system (1)

with mixed deterministic and stochastic uncertainty. The control cost function is defined by

$$J_N = \sum_{i=0}^{N-1} \|z_{i|k} - \tilde{x}_k\|_Q^2 + \|v_{i|k} - \tilde{u}_k\|_R^2 \quad (16)$$

$$+ \|z_{N|k} - \tilde{x}_k\|_P^2 + \|\bar{x}_k - \tilde{x}_k\|_T^2,$$

where $Q \geq 0$ is the state penalty matrix, $R > 0$ is the input penalty matrix, $P > 0$ is the terminal penalty matrix, and $(\tilde{x}_k, \tilde{u}_k)$ is an artificial steady state introduced to ensure feasibility of the optimization problem with respect to any setpoint s_k . The term $\|\bar{x}_k - \tilde{x}_k\|_T^2$ is incorporated into the cost function to penalize deviations between the desired steady state \bar{x}_k and the artificial steady state \tilde{x}_k for some weight matrix $T > 0$ [Alvarado et al., 2007].

At every sampling time k with a given $\theta_k = N_\theta s_k$ and measured states x_k , the proposed robust MPC approach involves solving the optimal control problem (OCP)

$$\min_{\mathbf{c}_k, \tilde{\theta}_k} J_N(\mathbf{c}_k, \tilde{\theta}_k; x_k, \theta_k) \quad (17a)$$

$$\text{s.t.: } (\tilde{x}_k, \tilde{u}_k) = M_\theta \tilde{\theta}_k, \quad (17b)$$

$$z_{0|k} = x_k, \quad (17c)$$

$$z_{i+1|k} = Az_{i|k} + Bv_{i|k}, \quad i = 0, \dots, N-1, \quad (17d)$$

$$v_{i|k} = Kz_{i|k} + c_{i|k}, \quad i = 0, \dots, N-1, \quad (17e)$$

$$z_{i|k} \in \mathbb{Z}_i, \quad v_{i|k} \in \mathbb{V}_i, \quad i = 0, \dots, N-1, \quad (17f)$$

$$(z_{N|k}, \tilde{\theta}_k) \in \mathbb{Z}_f^e, \quad (17g)$$

where $\mathbf{c}_k = (c_{0|k}, c_{1|k}, \dots, c_{N-1|k})$, \mathbb{Z}_i and \mathbb{V}_i are tightened constraint sets, and \mathbb{Z}_f^e is a terminal constraint set for the tracking problem. The MPC control law applied to (1) at time k is defined by

$$u_k = Kx_k + c_{0|k}^*(x_k, \theta_k), \quad (18)$$

where $c_{0|k}^*$ is the first element of the optimal control policy \mathbf{c}_k^* obtained by solving (17)

The OCP (17) is a convex quadratic program (QP) that can be solved efficiently to global optimality using standard optimization methods. Note that the controller is almost as simple to solve as a nominal MPC controller since the constraint tightening is done completely offline. The constraint tightening will be least conservative when the full uncertainty descriptions are taken into account.

Next, we discuss how the methods introduced in Section 3 can be used to select \mathbb{Z}_i and \mathbb{V}_i to ensure closed-loop satisfaction of constraints (2) and recursively feasibility of (17) as well as how to modify the target s_k to ensure offset-free control with respect to the mean of the system.

4.1 Infinite Cost

Using the dual mode scheme (6), the weight matrix P can be selected to represent exactly an infinite cost function. Combining (6) with the target calculator (7) results in $v_{i|k} - \tilde{u}_k = K(z_{i|k} - \tilde{x}_k)$ and $z_{i+1|k} - \tilde{x}_k = A(z_{i|k} - \tilde{x}_k) + B(v_{i|k} - \tilde{u}_k)$ for all $i \geq N$. The infinite tail of the cost function can then be computed as

$$\sum_{i=N}^{\infty} \|z_{i|k} - \tilde{x}_k\|_{Q+K^\top RK}^2 = \|z_{N|k} - \tilde{x}_k\|_P^2,$$

where $P > 0$ is the solution to $P - \Phi^\top P \Phi = Q + K^\top RK$.

4.2 State and Input Constraint Tightening

The sets \mathbb{Z}_i and \mathbb{V}_i can be computed using either robust tubes or probabilistic tubes (or a combination of them), as presented in Section 3. For example, the input constraint tightening can be done using the robust tubes while the state constraint tightening can be done using the combined robust-probabilistic tube method described in Proposition 2.

Robust tubes (10) yield constraints that are independent of the index i , which represents the number of steps predicted into the future from the current time k . In other words, $\mathbb{Z}_i = \mathbb{X} \ominus E$ and $\mathbb{V}_i = \mathbb{U} \ominus KE$ for all $i \in \mathbb{N}$. On the other hand, probabilistic tubes (11) and (15) are less conservative, but are a function of the future steps predicted i wherein $\mathbb{Z}_i = \{z : Hz \leq f - \eta_i\}$ and $\mathbb{V}_i = \{v : Gv \leq g - \mu_i\}$. Note, however, that these sets can be computed exactly offline for $i = 0, \dots, N-1$.

4.3 Terminal Constraint Set

The terminal constraint set \mathbb{Z}_f^e is used to ensure that the state and input constraints are satisfied over the infinite prediction horizon using the dual mode prediction paradigm in (6). Constraints are invoked explicitly in mode 1, while they are enforced implicitly in mode 2 using \mathbb{Z}_f^e . In mode 2, constraints (11) and (15) can be written as

$$z_{N+j|k} \in \mathbb{Z}_{N+j}, \quad v_{N+j|k} \in \mathbb{V}_{N+j}, \quad j \in \mathbb{N}_0, \quad (19)$$

where

$$z_{N+j|k} = \tilde{x}_k + \Phi^j(z_{N|k} - \tilde{x}_k),$$

$$v_{N+j|k} = \tilde{u}_k + K\Phi^j(z_{N|k} - \tilde{x}_k).$$

Clearly, the future predicted system behavior in mode 2 is only a function of $z_{N|k}$ and $\tilde{\theta}_k$, so that we can define the maximal admissible set \mathbb{Z}_∞^e as the set of all $(z_{N|k}, \tilde{\theta}_k)$ such that (19) holds for all future steps $j \in \mathbb{N}_0$.

When using robust tubes, the computation of \mathbb{Z}_∞^e can be simplified by defining an invariant set for tracking by taking advantage of the fact that the constraints are independent of future time index j [Limón et al., 2008].

When using probabilistic tubes for fulfilling (19), however, there are an infinite number of time-dependent constraints that define \mathbb{Z}_∞^e . In this case, an inner approximation of \mathbb{Z}_∞^e can be defined by replacing η_{N+j} and μ_{N+j} by bounds $\bar{\eta}$ and $\bar{\mu}$ beyond a finite horizon $j = \hat{N}$ [Kouvaritakis et al., 2010]. The bounds $\bar{\eta}$ and $\bar{\mu}$ can be straightforwardly derived using the methods discussed in Section 3. This leads to a set $\mathbb{Z}_{\hat{N}}^e \subset \mathbb{Z}_\infty^e$. Even though the set $\mathbb{Z}_{\hat{N}}^e$ is defined by an infinite number of inequalities, only the first $\hat{N} + N^*$ inequalities are needed to construct $\mathbb{Z}_{\hat{N}}^e$ provided that N^* is sufficiently large [Gilbert and Tan, 1991]. The smallest allowable N^* can be found offline by solving a sequence of linear programs. Hence, setting $\mathbb{Z}_f^e = \mathbb{Z}_{\hat{N}}^e$ guarantees satisfaction of state and input constraints in mode 2.

4.4 Offset-Free Tracking through Disturbance Estimation

If the uncertainty d_k tends to a steady-state value d_∞ , the output of the closed-loop system, under the control law (4), will be offset from the desired asymptotic setpoint

s_∞ . Due to persistent excitation of stochastic disturbances, the closed-loop system cannot converge asymptotically to any point. However, the offset with respect to the average/mean value of (1) can be eliminated so that the system “oscillates” with bounded variance around s_∞ .

Proposition 3. Assume that the deterministic disturbances d_k and reference s_k are asymptotically constant, $\rho(\Phi) < 1$ with K being the optimal control law for unconstrained MPC, and the target calculator (7) has a unique solution for all s_k . Assume that the OCP (17) is unconstrained for all future $t \geq j$ for some finite $j \in \mathbb{N}_0$ and the mean of the closed-loop system is convergent. Then $m_\infty^y = s_\infty + (C + DK)(I - \Phi)^{-1}d_\infty$, where $m_k^y = \mathbb{E}[y_k]$ is the mean of the output at time k .

Proof: Let $m_k^x = \mathbb{E}[x_k]$ and $m_k^u = \mathbb{E}[u_k]$ denote the mean of the closed-loop states and inputs at time k , respectively. The converged mean of (1) must satisfy

$$m_\infty^x = Am_\infty^x + Bm_\infty^u + d_\infty, \quad m_\infty^y = Cm_\infty^x + Dm_\infty^u.$$

The converged target calculator must satisfy

$$\bar{x}_\infty = A\bar{x}_\infty + B\bar{u}_\infty, \quad s_\infty = C\bar{x}_\infty + D\bar{u}_\infty.$$

When (17) is unconstrained, the input is equal to $u_k = \bar{u}_k + K(x_k - \bar{x}_k)$, where K is the static feedback gain. The mean of the converged closed-loop input must then satisfy $m_\infty^u = \bar{u}_\infty + K(m_\infty^x - \bar{x}_\infty)$. Subtracting the mean from the target and substituting the control input results in

$$\begin{aligned} m_\infty^x - \bar{x}_\infty &= \Phi(m_\infty^x - \bar{x}_\infty) + d_\infty, \\ m_\infty^y - s_\infty &= (C + DK)(m_\infty^x - \bar{x}_\infty). \end{aligned}$$

Solving for $(m_\infty^x - \bar{x}_\infty)$ and substituting into the latter equation directly proves the claim. \square

Proposition 3 provides an explicit expression for the offset in the mean value of the outputs. The setpoint s_k in the nominal tracking problem (7) can now be modified to asymptotically eliminate this offset. Let s_k^{des} be the desired setpoint for the system. Define

$$s_k = s_k^{\text{des}} - (C + DK)(I - \Phi)^{-1}\hat{d}_k, \quad (20)$$

where \hat{d}_k is an estimate of the disturbances at current time k that can be calculated by

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{bmatrix} = \begin{bmatrix} A & I \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{d}_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + \begin{bmatrix} L_x \\ L_d \end{bmatrix} (x_k - \hat{x}_k), \quad (21)$$

or using a simpler filter of the form

$$\hat{d}_k = \lambda_f \hat{d}_{k-1} + (1 - \lambda_f)(x_k - Ax_{k-1} - Bu_{k-1}). \quad (22)$$

There exist observer gains L_x and L_d such that the estimator (21) converges (in the mean) when the augmented system satisfies standard observability conditions [Pannocchia and Rawlings, 2003, Maeder et al., 2009]. The gains L_x and L_d can be designed optimally using a Kalman filter, whereas the simple filter (22) ensures that the mean $\mathbb{E}[\hat{d}_k]$ converges to d_∞ at a rate fixed by filter coefficient λ_f .

4.5 Recursive Feasibility

The existence of a solution \mathbf{c}_k^* to the OCP (17) at any time k ensures that the state and input constraints are satisfied in the future with a certain probability, but not for all possible realizations of disturbances (unless the robust tubes (10) are used to enforce constraints or ϵ_j is set to zero in (2a)). It is known that the probability of constraint violation j steps ahead into the future at

time k is not the same as that in $j - 1$ steps into the future at time $k + 1$ due to the realization of states x_{k+1} (that is unknown at time k). To address this issue, probabilistic tubes that are recursively feasible have been proposed in [Kouvaritakis et al., 2010]. The key idea is to further tighten the constraint back-off parameters η_i and μ_i based on the fact that the states will be realized in the future. This approach can be used to modify the tightened state and input constraint sets \mathbb{Z}_i and \mathbb{V}_i to guarantee recursive feasibility of the OCP (17). Readers are referred to Theorem 3 of [Kouvaritakis et al., 2010] for explicit expressions for the back-off parameters, and the subsequent bounds on these parameters which can be used when defining the terminal set Z_f^e .

5. NUMERICAL EXAMPLE

We demonstrate the advantages of separately accounting for the plant-model mismatch and stochastic exogenous disturbances in MPC using our proposed method. The example system is based on the DC-DC converter taken from [Cannon et al., 2011, Lorenzen et al., 2015]. The system has the form (1) with

$$A = \begin{bmatrix} 0.98 & 0.0075 \\ -0.143 & 0.976 \end{bmatrix}, \quad B = \begin{bmatrix} 4.778 \\ 0.115 \end{bmatrix}, \quad C = [0 \ 1], \quad D = 0,$$

where d_k is a plant-model mismatch term exactly given by

$$d_k = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix} x_k + \begin{bmatrix} 0.02 \\ 0 \end{bmatrix} u_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} p_k.$$

The first two terms represent mismatch in the A and B matrices while p_k is an unknown persistent disturbance

$$p_k = \begin{cases} 0, & k < 5, \\ 0.1, & k \geq 5. \end{cases}$$

The expression for d_k is unknown to the controller, however, it can be bounded in a set \mathbb{D} based on comparisons of the model to data. The distribution of the stochastic disturbance w_k is assumed to be a Gaussian distribution with covariance $0.2I$ truncated at $\mathbb{W} = \{w : \|w\|_\infty \leq 0.1\}$.

The weights in the MPC cost function are $Q = \text{diag}(1, 10)$ and $R = 1$, and the prediction horizon is $N = 5$. The gain K is chosen to be the solution to the unconstrained linear quadratic regulator (LQR). The initial state in all simulations was chosen to be $x_0 = [2.5, 2.8]^\top$.

Two individual state chance constraints of the form (2a) are considered

$$\mathbb{P}[x_{1,k} \leq 2] \geq 0.8, \quad \mathbb{P}[x_{2,k} \leq 3] \geq 0.8.$$

The hard input constraints are given by

$$\mathbb{U} = \{u : \|u\|_\infty \leq 0.8\}.$$

The performance of the proposed robust MPC approach (the OCP (17) with Proposition 2 used to tighten the state chance constraints) was evaluated by a Monte Carlo simulation using 100 realizations. Simulation results for the closed-loop system are shown in Fig. 1. We observed 0% constraint violation in the first few steps due to minimal mismatch between the true plant and the model. After the persistent disturbance kicks in at step 5, an average of 12% constraint violation is observed over the next few steps. This is reasonably close to the maximum allowed constraint violation level 20%, suggesting minimal conservatism of the proposed robust MPC approach. This

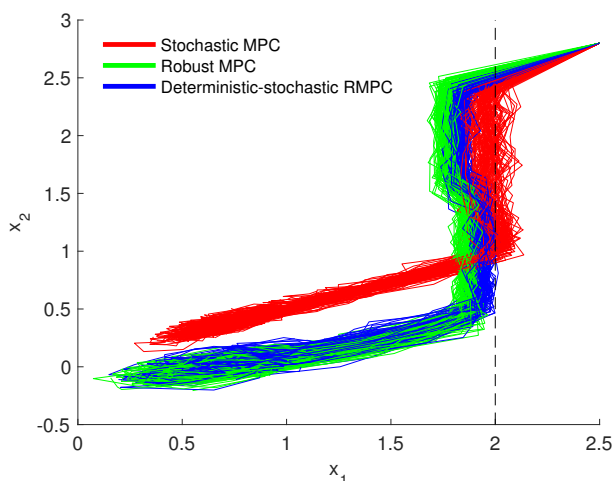


Fig. 1. Comparison of closed-loop system responses for various MPC approaches under 100 uncertainty realizations (x_1 and x_2 denote the system states).

is due to the fact that the constraints were tightened using knowledge of the distribution of w_k . Since there is one control input in the system, only one output can be tracked without offset due to the rank condition of the target calculator (7). Using the setpoint update rule (20), we are able to ensure the mean of the output $y = x_2$ converges to the origin as desired.

For comparison, we provide closed-loop simulation results using a robust tube-based MPC approach for tracking [Alvarado et al., 2007] and a stochastic MPC approach [Lorenzen et al., 2015] (see Fig. 1). As expected, the robust MPC approach ensures 0% constraint violation at all times, which makes the results more conservative than that of the proposed approach. On the other hand, the stochastic MPC approach does not account for the plant-model mismatch, resulting in an average of 95% constraint violation in the first 8 steps. Also, the stochastic MPC approach does not update the target setpoint resulting in offset in the mean of the output at steady state.

6. CONCLUSIONS AND FUTURE WORK

This paper presents a robust MPC approach for offset-free tracking of linear systems in the presence of bounded deterministic and stochastic disturbances. The deterministic disturbance accounts for structural/parameter mismatch between the plant and model while the stochastic disturbance represents exogenous disturbances that vary randomly with some known distribution. Probabilistic/robust constraint satisfaction is guaranteed by tightening the constraints offline using well-known tube methods, while offset-free behavior is ensured using a disturbance estimator.

Future work will involve exploring new methods for ensuring stability and feasibility. An example is provided in [Lorenzen et al., 2015] wherein a first-step constraint is added to ensure recursively feasibility, which leads to an increase in the domain of attraction of the MPC problem. Performance and constraint satisfaction can be improved by optimizing the feedback gain online. An affine disturbance feedback parametrization [Goulart et al., 2006] would ensure the MPC problem remains convex.

Future work should also be done on demonstrating the flexibility and usefulness of the new mixed deterministic-stochastic system description. This includes illustrating how the disturbance sets \mathbb{D} and \mathbb{W} and the distribution of W can be approximated from data for real-life systems, as well as showing improvements in closed-loop performance from treating these two uncertainty sources separately.

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